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Brief paper

Distributed output feedback control of Markov jump multi-agent systems[☆]

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ABSTRACT

In this paper, distributed output feedback control of Markov jump multi-agent systems (MASs) is investigated. Both dynamic equations and index functions of the MASs involved contain Markov jump parameters. The information available for each agent to design their controllers are only the noisy output and the jump parameters. By Markov jump optimal filtering theory and the mean field approach, distributed output feedback control laws are presented. Under some mild conditions, it is shown that the closed-loop system is uniformly stable and the distributed control is sub-optimal.

1. Introduction

Recently, the control and optimization problem of multi-agent systems (MASs) has become a hot topic in the systems and control community. One key concerned issue is how to design distributed control laws based on agents' local information. For the case of large population, the main difficulty lies in the high computational complexity. To overcome this difficulty, the mean field (MF) approach is extended and applied (Huang, Caines, & Malhame, 2003; Huang, Caines, & Malhamé, 2007; Li & Zhang, 2008a,b). Especially, Huang et al. (2003, 2007) developed the Nash certainty equivalence methodology based on the MF theory, with which distributed ε-Nash equilibrium strategies were given for the games of large population MASs coupled via discounted costs. Li and Zhang (2008a,b) considered the case where agents are coupled via their stochastic long run time-average indices, and obtained asymptotical Nash equilibria in the probabilistic sense.

Uncertainty is hard to avoid in practice, for instance, failure of system components and environmental uncertainties. As a proper mathematical model to describe the dynamical behaviors of the systems in an environment with abrupt changes, Markov jump systems have been studied for many years (Costa, Fragoso, & Marques, 2005; Mariton, 1990; Sworder, 1969; Wonham, 1970). Recently, Wang and Zhang (2012a,b) investigated MF games of Markov jump MASs.

Another issue worthy of consideration is the case where only partial observation of system states can be obtained, since, strictly speaking, all practical control problems are based on the information of output measurements instead of the full states. The control problems based on the full state information, no matter what it is, deterministic or stochastic, can only be an approximation of real problems. For output feedback control of conventional systems (i.e. the case where only one agent is considered), readers are referred to Bensoussan (1992), Davis (1977) and Wonham (1968). Huang, Caines, and Malhamé (2006) considered continuous-time MF games for MASs with time-invariant parameters based on output measurements and provided a set of distributed control laws.

In this paper, we investigate distributed output feedback control of discrete-time MASs under the game-theoretic framework. Both dynamic equations and index functions of the MASs involved contain Markov jump parameters. The information available for each agent is the noisy output and the Markov jump parameters instead of its state. Compared with the previous works (Wang & Zhang, 2012a,b), here index functions are more general, in which Markov jump parameters are allowed.

For this kind of systems, we design distributed control by filtering theory and the MF approach. Due to partial observation of

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the system states, each agent needs to estimate both its own state and the population state average under stochastic disturbances. Different from the case of time-invariant parameters (Huang et al., 2006), not only agents' estimates for their own states are dynamic systems driven by a family of independent Markov chains, but also due to agents' indices containing Markov parameters agents' MF estimation is a function of transient distributions of Markov chains, which brings much difficulty for analysis of the closed-loop system. We first prove the MF estimation function is bounded by using the ergodicity of Markov parameters and the properties of matrix norms, and then get uniform stability of the closed-loop system. By exploiting the independence of agents' Markov parameters and analyzing the estimation errors of Markov jump optimal filters, we use the MF estimation function to approximate the population state average, and obtain a sub-optimal distributed control.

The following notations will be used in the paper. For a given vector or matrix X, $\|X\|$ denotes the Euclidean vector norm or the matrix norm induced by the Euclidean vector norm of X. For a family of $\mathbb{R}^n$-valued r.v.s $\{\xi_\lambda, \lambda \in \Lambda\}$, $\sigma(\xi_\lambda, \lambda \in \Lambda)$ denotes the σ algebra $\sigma\{[\xi_\lambda \in B], B \in \mathcal{B}^n, \lambda \in \Lambda\}$, where $\mathcal{B}^n$ is an n dimensional Borel σ algebra. For a sequence of real numbers $\{a_n, n = 0, 1, \dots\}$ and a sequence of positive numbers $\{b_n, n = 0, 1, \dots\}$, $a_n = O(b_n)$ denotes $\limsup_{n\to\infty} |a_n|/b_n < \infty$.

2. Problem description

Consider the following MAS:

$$x_i(k+1) = A_{\theta_i(k)}x_i(k) + u_i(k) + D_{\theta_i(k)}w_i(k+1), \tag{1}$$

$$y_i(k) = C_{\theta_i(k)}x_i(k) + L_{\theta_i(k)}v_i(k), \quad 1 \le i \le N, \tag{2}$$

where $x_i, u_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^p$ are the state, input and output of the agent i, respectively; $\{w_i, 1 \le i \le N\}$ and $\{v_i, 1 \le i \le N\}$ are stochastic noises. $\{\theta_i(k)\}$ is a family of discrete-time ergodic Markov chains taking value in $S = \{1, 2, \dots, m\}$ with the same transition probability matrix $P = \{p_{ij}, i, j = 1, \dots, m\}$ and stationary distribution $\pi = \{\pi_j, j = 1, \dots, m\}$. The indices of N agents are described by

$$J_i^N(u) = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T} E\left\|x_i(k+1) - H_{\theta_i(k)}x^{(N)}(k) - \alpha_{\theta_i(k)}\right\|^2,$$
$$1 \le i \le N, \tag{3}$$

where $H(\cdot) \in \mathbb{R}^{n\times n}$, $\alpha(\cdot) \in \mathbb{R}^n$, $x^{(N)}(k) = \sum_{i=1}^{N} x_i(k)/N$ is population state average, and $u = (u_1, \dots, u_N)$.

Remark 2.1. The above model has a wide practical background. Consider a market composed of many firms. For the firm i, the profit level x_i depends on the input u_i and the jump parameter θ_i which depicts operational uncertainty, such as failure in the production line. Assume that, the profit goal for each firm is to attain some function of the average profit of all firms and the jump parameter.

The objective of this paper is to design distributed output feedback control laws for agents in the system (1)–(3) under the game-theoretic framework, that is, to optimize the index group (3) over the group of control sets:

$$\mathcal{U}_{l,i} = \left\{u | u(k) \in \sigma\{y_i(j), \theta_i(j), 0 \le j \le k\}\right\}, \quad 1 \le i \le N.$$

For the convenience of citation, here we list the main assumptions to be used in the paper:

(A1) $\{w_i(k), 1 \le i \le N\}$ and $\{v_i(k), 1 \le i \le N\}$ are d-dimensional Gaussian white noise sequences defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, $E[w_i(k)] = E[v_i(k)] = 0$, $E[w_i(k)w_i^T(j)] = E[v_i(k)v_i^T(j)] = \delta_{kj}I_d$, $1 \le i \le N$, where $\delta_{kj} = \begin{cases} 1, & \text{if } k = j; \\ 0, & \text{otherwise.}\end{cases}$

(A2) (A, D, P) is mean-square stabilizable, and (A, C, P) is mean-square detectable,[2] where $A = (A_1, \dots, A_m)$, $C = (C_1, \dots, C_m)$, and $D = (D_1, \dots, D_m)$. $L_jL_j^T, j = 1, \dots, m$ are positive definite.

(A3) Initial values $\{x_{i0}\}$ are independent random variables; $Ex_{i0} = x_0$, $1 \le i \le N$; $\max_{1\le i\le N} E\|x_{i0}\|^2 < \infty$.

3. Design of control law

To inspire the design of distributed control, we first consider the case where all the system states are known. Let

$$\mathcal{U}_{g,i} = \left\{u | u(k) \in \sigma\left\{\bigcup_{1\le i\le N}\sigma(x_i(j), \theta_i(j), 0 \le j \le k)\right\}\right\}.$$

Theorem 3.1. *For the system (1) and the index (3), if Assumptions (A1) and (A2) hold, then for any set of control $\{u_i \in \mathcal{U}_{g,i}, 1 \le i \le N\}$,*

$$J_i^N(u) \ge \sum_{j=1}^{m} \pi_j tr(D_jD_j^T), \quad 1 \le i \le N. \tag{4}$$

In particular, if we take $u_i(k) = H_{\theta_i(k)}x^{(N)}(k) + \alpha_{\theta_i(k)} - A_{\theta_i(k)}x_i(k)$, then

$$J_i^N(u) = \sum_{j=1}^{m} \pi_j tr(D_jD_j^T), \quad 1 \le i \le N. \tag{5}$$

Proof. From (A1) and (3) it follows that

$$J_i^N(u) = \limsup_{T\to\infty} \frac{1}{T}\sum_{k=0}^{T}\Big\{E\left\|A_{\theta_i(k)}x_i(k) + u_i(k) - H_{\theta_i(k)}x^{(N)}(k) - \alpha_{\theta_i(k)}\right\|^2 + E\|D_{\theta_i(k)}w_i(k+1)\|^2\Big\}. \tag{6}$$

By the ergodicity of $\{\theta_i(k)\}$,

$$\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\|D_{\theta_i(k)}w_i(k+1)\|^2 = \sum_{j=1}^{m}\limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} tr(D_j^TD_j)p_j(k) = \sum_{j=1}^{m}\pi_j tr(D_j^TD_j),$$

where $p_j(k) = \mathcal{P}(\theta_1(k) = j)$. This together with (6) gives that

$$J_i^N(u) \ge \limsup_{T\to\infty}\frac{1}{T}\sum_{k=0}^{T} E\left\|A_{\theta_i(k)}x_i(k) + u_i(k) - H_{\theta_i(k)}x^{(N)}(k) - \alpha_{\theta_i(k)}\right\|^2 + \sum_{j=1}^{m}\pi_j tr(D_j^TD_j)$$
$$\ge \sum_{j=1}^{m}\pi_j tr(D_j^TD_j), \quad 1 \le i \le N. \tag{7}$$

Thus, by taking $u_i(k) = H_{\theta_i(k)}x^{(N)}(k) + \alpha_{\theta_i(k)} - A_{\theta_i(k)}x_i(k)$, one can get $J_i(u) = \sum_{j=1}^{m}\pi_j tr(D_j^TD_j)$. □

[2] (A, D, P) is mean-square stabilizable, if there is a feedback law $u(k) = F_{\theta_1(k)}(k)x(k)$, such that $x(t) = A_{\theta_1(k)}(k)x(k) + D_{\theta_1(k)}(k)u(k)$ is mean-square stable, i.e., $E\|x(k)\|^2 \to 0$. (A, C, P) is mean-square detectable, if (A^T, C^T, P^T) is mean-square stabilizable. For the details, the readers are referred to Costa et al. (2005).

Before designing distributed output feedback control, we first give distributed control laws for the case where the local state is available to each agent.

Let $g(k)$ denote the estimate of $x^{(N)}(k)$. Then, by Theorem 3.1 we get the following control laws: for $1 \le i \le N$,

$$u_i(k) = -A_{\theta_i(k)}x_i(k) + H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)}. \tag{8}$$

Applying the control laws into (1) gives

$$x_i(k+1) = H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)} + D_{\theta_i(k)}w_i(k+1),$$

which leads to

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N x_i(k+1) &= \frac{1}{N}\sum_{i=1}^N H_{\theta_i(k)}g(k) + \frac{1}{N}\sum_{i=1}^N \alpha_{\theta_i(k)}\\ &\quad + \frac{1}{N}\sum_{i=1}^N D_{\theta_i(k)}w_i(k+1).\end{aligned} \tag{9}$$

Let $N \to \infty$. Noticing $\theta_i(k)$, $1 \le i \le N$ are independent of each other, by the MF approach,[3] the estimation function $g(k)$ for $x^{(N)}(k)$ in the large-population system (1)–(3) should satisfy the following recursive equation:

$$g(k+1) = \sum_{j=1}^m p_j(k)(H_j g(k) + \alpha_j), \qquad g(0) = x_0. \tag{10}$$

We now design distributed control laws for the case where only the noisy output and the jump parameters are available.

From the results in Chizeck and Ji (1990) and Costa and Tuesta (2003), the Markov jump optimal filtering equation for the system (1)–(2) is

$$\begin{aligned}\hat{x}_i(k+1) &= A_{\theta_i(k)}\hat{x}_i(k) + u_i(k) + M_{\theta_i(k)}\\ &\quad\times [y_i(k) - C_{\theta_i(k)}(A_{\theta_i(k)}\hat{x}_i(k) + u_i(k))], \qquad \hat{x}_i(0) = x_0,\end{aligned} \tag{11}$$

where $1 \le i \le N$,

$$M_j = Y_j C_j^T (L_j L_j^T \pi_j + C_j Y_j C_j^T)^{-1}, \tag{12}$$

$$\begin{aligned}Y_k \;=\; &\sum_{j=1}^m p_{jk}\big[A_j Y_j A_j^T + \pi_j D_j D_j^T\\ &- A_j Y_j C_j^T (L_j L_j^T \pi_j + C_i Y_j C_j^T)^{-1} C_j Y_j A_j^T\big],\end{aligned} \tag{13}$$

$$Z_j = Y_j - M_j C_j Y_j, \quad j, k = 1, \ldots, m. \tag{14}$$

Remark 3.1. The filter (11)–(14) was first given in Chizeck and Ji (1990). Costa and Tuesta (2003) proved that (11)–(14) was optimal in the class of linear Markov jump filters. Since $\theta_i(k)$ is known in (1)–(2), the best linear estimator of $x_i(k)$ is the Kalman filter for time-varying systems. However, as pointed out by Chizeck and Ji (1990), the off-line computation load of this kind of Kalman filter will grow exponentially as time goes on. On the other hand, gains of the filter (11)–(14) are time-invariant and with much lower computation load.

Substituting the filtering $\hat{x}_i$ for the local state x_i, by (8) we obtain the following distributed output feedback control laws: for $1 \le i \le N$,

$$u_i(k) = H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)} - A_{\theta_i(k)}\hat{x}_i(k), \tag{15}$$

where $g(k)$ and $\hat{x}_i(k)$ satisfy (10) and (11), respectively.

4. Property of the closed-loop system

To ensure the stability of the closed-loop system, assume:

(A4) $\sum_{j=1}^m \pi_j H_j$ is stable, i.e., all its eigenvalues are within the unit circle.

[3] The key idea of MF approaches is to replace the sum of effects of all agents to one by the aggregate effect (Huang et al., 2007; Lasry & Lions, 2007; Weintraub, Benkard, & Van Roy, 2008).

Lemma 4.1. *Under* (A4), *for* (1)–(3) *we have*

$$\sup_{k\ge 0} \|g(k)\|^2 < \infty. \tag{16}$$

Proof. From (10) it follows that

$$g(k+1) = \prod_{l=0}^k H^{p(l)}x_0 + \sum_{l=0}^k \prod_{s=l+1}^k H^{p(s)}\alpha^{p(l)}, \tag{17}$$

where $H^{p(l)} = \sum_{j=1}^m p_j(l)H_j$, $\alpha^{p(l)} = \sum_{j=1}^m p_j(l)\alpha_j$. From (A4) and Horn and Johnson (1990), there exists a matrix norm $\|\cdot\|_0$ induced by the vector norm $\|\cdot\|_0$ such that $\|\sum_{j=1}^m \pi_j H_j\|_0 = \rho < 1$. Since $\theta_i(k)$ is ergodic, there exists l_0, such that for all $l \ge l_0$,

$$\max_{1\le j\le m}\|p_j(l) - \pi_j\|_0 \le \frac{1-\rho}{2\sum_{j=1}^m \|H_j\|_0}.$$

Thus, for all $l \ge l_0$,

$$\begin{aligned}\left\|H^{p(l)}\right\|_0 &\le \left\|\sum_{j=1}^m \pi_j H_j\right\|_0 + \left\|\sum_{j=1}^m (p_j(l) - \pi_j)H_j\right\|_0\\ &\le \rho + \frac{(1-\rho)\sum_{j=1}^m \|H_j\|_0}{2\sum_{j=1}^m \|H_j\|_0} \triangleq \rho_0 < 1.\end{aligned}$$

By a straightforward calculation, we have for all $k \ge l_0$,

$$\begin{aligned}\left\|\prod_{l=0}^k H^{p(l)}\right\|_0 &\le \prod_{l=0}^{l_0}\left\|H^{p(l)}\right\|_0 \times \prod_{l=l_0+1}^k \left\|H^{p(l)}\right\|_0\\ &\le \rho_0^{k-l_0}\prod_{l=0}^{l_0}\left\|H^{p(l)}\right\|_0 \le \prod_{l=0}^{l_0}\left\|H^{p(l)}\right\|_0\end{aligned} \tag{18}$$

and

$$\begin{aligned}\left\|\sum_{l=0}^k \prod_{s=l+1}^k H^{p(s)}\alpha^{p(l)}\right\|_0 &\le \left(\sum_{l=0}^{l_0-1} + \sum_{l=l_0}^k\right)\left\|\prod_{s=l+1}^k H^{p(s)}\right\|_0 \left\|\alpha^{p(l)}\right\|_0\\ &\le \sum_{l=0}^{l_0-1}\left\|\prod_{s=l+1}^{l_0} H^{p(s)}\right\|_0 \max_{1\le j\le m}\left\|\alpha_j\right\|_0 + \frac{1}{1-\rho_0}\max_{1\le j\le m}\left\|\alpha_j\right\|_0 \triangleq c_0.\end{aligned}$$

This together with (17) and (18) implies for all $k \ge l_0$,

$$\|g(k+1)\|_0^2 \le 2\prod_{l=0}^{l_0}\left\|H^{p(l)}\right\|_0^2 \|x_0\|_0^2 + 2c_0^2 < \infty.$$

Thus, from equivalence of norms in a finite dimensional space, one can get (16). □

We now show uniform stability of the closed-loop system.

Theorem 4.1. *If Assumptions* (A1)–(A4) *hold, then for the system* (1)–(3) *under* (15), *we have*

$$\max_{1\le i\le N}\limsup_{T\to\infty}\frac{1}{T}\sum_{k=1}^T E\big[\|x_i(k)\|^2 + \|u_i(k)\|^2\big] < \infty. \tag{19}$$

Proof. Applying (15) into (1) leads to that

$$x_i(k+1) = A_{\theta_i(k)}\tilde{x}_i(k) + H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)} + D_{\theta_i(k)}w_i(k+1), \tag{20}$$

where $\tilde{x}_i(k) \triangleq x_i(k) - \hat{x}_i(k)$. From (A2) together with Costa et al. (2005) it follows that for $j = 1, \ldots, m$,

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\|\tilde{x}_i(k) I_{[\theta_i(k)=j]}\|^2 = tr(Z_j), \tag{21}$$

where Z_j satisfies (13). Noticing

$$E\left[\tilde{x}_i(k)\tilde{x}_i^T(k) I_{[\theta_i(k)=j]} I_{[\theta_i(k)=l]}\right] = 0, \quad j \neq l,$$

we have

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\|A_{\theta_i(k)}\tilde{x}_i(k)\|^2 = \sum_{j=1}^{m} tr(A_j Z_j A_j^T). \tag{22}$$

From (A1) and the ergodicity of $\{\theta_i(k)\}$, we have

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\|x_i(k+1)\|^2 \\
&= \limsup_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\Big\{\|A_{\theta_i(k)}\tilde{x}_i(k) + H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)}\|^2 \\
&\quad + \|D_{\theta_i(k)}w_i(k+1)\|^2\Big\} \\
&\leq \sum_{j=1}^{m} 3tr(A_j Z_j A_j^T) + \sum_{j=1}^{m} \pi_j \Big[3tr(H_j H_j^T) \sup_{k\geq 0} \|g(k)\|^2 \\
&\quad + 3\|\alpha_j\|^2 + tr(D_j D_j^T)\Big].
\end{aligned} \tag{23}$$

Substituting (15) into (11) together with (2), we have

$$\begin{aligned}
\hat{x}_i(k+1) &= H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)} + M_{\theta_i(k)}\big[C_{\theta_i(k)}(A_{\theta_i(k)}\tilde{x}_i(k) \\
&\quad + C_{\theta_i(k)}D_{\theta_i(k)}w_i(k+1)) + L_{\theta_i(k)}v_i(k+1)\big].
\end{aligned} \tag{24}$$

This together with the ergodicity of $\{\theta_i(k)\}$ gives

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} E\left\|\hat{x}_i(k+1)\right\|^2 \\
&\leq \sum_{j=1}^{m} \pi_j \Big[4\Big(\|H_j\|^2 \sup_k \|g(k)\|^2 + \|\alpha_j\|^2\Big) \\
&\quad + 2tr(M_j C_j (A_j Z_j A_j^T + D_j D_j^T) C_j^T M_j^T) + 2tr(M_j L_j L_j^T M_j^T)\Big] \triangleq c_1.
\end{aligned}$$

By (15), it follows that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} E\left\|u_i(k)\right\|^2 \\
&\leq \sum_{j=1}^{m} \pi_j \Big[4\|H_j\|^2 \sup_k \|g(k)\|^2 + 4\|\alpha\|^2 + 2c_1\|A_j\|^2\Big].
\end{aligned}$$

This together with (23) gives the theorem. □

To analyze the optimality of the control laws (15), we first give the approximation error of the MF estimation g to $x^{(N)}$ in the mean-square sense.

Theorem 4.2. *For (1)–(3), if (A1)–(A4) hold, then under (15),*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\|x^{(N)}(k) - g(k)\|^2 = O\left(\frac{1}{N}\right). \tag{25}$$

Proof. From (20) and (24), it follows that

$$\begin{aligned}
\tilde{x}_i(k+1) &= A_{\theta_i(k)}\tilde{x}_i(k) - M_{\theta_i(k)}\big[C_{\theta_i(k)}(A_{\theta_i(k)}\tilde{x}_i(k) \\
&\quad + D_{\theta_i(k)}w_i(k+1)) + L_{\theta_i(k)}v_i(k+1)\big] + D_{\theta_i(k)}w_i(k+1).
\end{aligned} \tag{26}$$

Noticing that $\{\tilde{x}_i(k)\}$, $1 \leq i \leq N$ are independent of each other, from $E\tilde{x}_i(k) = 0$ and (21) it follows that

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\|x^{(N)}(k) - \hat{x}^{(N)}(k)\|^2 \\
&= \lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T} E\left\|\frac{1}{N}\sum_{i=1}^{N} \tilde{x}_i(k)\right\|^2 = \frac{1}{N}\sum_{j=1}^{m} tr(Z_j),
\end{aligned} \tag{27}$$

where $\hat{x}^{(N)}(k) = \frac{1}{N}\sum_{i=1}^{N} \hat{x}_i(k)$. By (24) we have

$$\begin{aligned}
\hat{x}^{(N)}(k+1) &= \frac{1}{N}\sum_{i=1}^{N}\big[H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)}\big] + \frac{1}{N}\sum_{i=1}^{N} M_{\theta_i(k)} \\
&\quad \times \big[C_{\theta_i(k)}(A_{\theta_i(k)}\tilde{x}_i(k) + D_{\theta_i(k)}w_i(k+1)) + L_{\theta_i(k)}v_i(k+1)\big].
\end{aligned} \tag{28}$$

From (26), (A1) and (A3), we can get

$$E[\tilde{x}_i(k)w_i^T(k+1)] = 0, \qquad E[\tilde{x}_i(k)v_i^T(k+1)] = 0, \quad k \geq 0.$$

This together with (28) gives

$$\begin{aligned}
&\lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} E\left\|\hat{x}^{(N)}(k+1) - \frac{1}{N}\sum_{i=1}^{N}\big[H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)}\big]\right\|^2 \\
&= \frac{N}{N^2} \lim_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} E\Bigg\|\sum_{i=1}^{N} M_{\theta_i(k)}\big[C_{\theta_i(k)}(A_{\theta_i(k)}\tilde{x}_i(k) \\
&\quad + D_{\theta_i(k)}w_i(k+1)) + L_{\theta_i(k)}v_i(k+1)\big]\Bigg\|^2 \\
&= \frac{1}{N}\sum_{j=1}^{m} \pi_j\big[tr(M_j C_j (A_j Z_j A_j^T \\
&\quad + D_j D_j^T) C_j^T M_j^T) + tr(M_j L_j L_j^T M_j^T)\big].
\end{aligned} \tag{29}$$

From $E\big[H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)}\big] = g(k+1)$, it follows that

$$\begin{aligned}
&E\left\|\frac{1}{N}\sum_{i=1}^{N}\big[H_{\theta_i(k)}g(k) + \alpha_{\theta_i(k)}\big] - g(k+1)\right\|^2 \\
&\leq \frac{2}{N}\left[\max_{1\leq j\leq m} \|H_j\|^2 \sup_{k\geq 0} \|g(k)\|^2 + \max_{1\leq j\leq m} \|\alpha_j\|^2\right],
\end{aligned} \tag{30}$$

which together with (27) and (29) implies that

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{k=1}^{T} E\left\|x^{(N)}(k+1) - g(k+1)\right\|^2 \leq O\left(\frac{1}{N}\right). \quad \square$$

Now we are in a position to show the sub-optimality of the distributed control (15).

Theorem 4.3. *For the system (1)–(3), if (A1)–(A4) hold, then under the control laws (15), we have*

$$J_i^N(u) = \sum_{j=1}^{m} \pi_j tr(D_j D_j^T) + \sum_{j=1}^{m} tr(A_j Z_j A_j^T) + O\left(\frac{1}{\sqrt{N}}\right). \tag{31}$$

Proof. From (3) and (20) and (A1) it follows that

$$J_i^N(u) = \lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\Big\{E\left\|A_{\theta_i(k)}\tilde{x}_i(k)\right\|^2 + 2E\big[\tilde{x}_i^T(k)A_{\theta_i(k)}^T H_{\theta_i(k)}(g(k)-x^{(N)}(k))\big] + E\left\|H_{\theta_i(k)}[g(k)-x^{(N)}(k)]\right\|^2\Big\} + \sum_{j=1}^{m}\pi_j tr(D_jD_j^T). \quad (32)$$

Noticing

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}E\Big\{\tilde{x}_i^T(k)A_{\theta_i(k)}^T H_{\theta_i(k)}[g(k)-x^{(N)}(k)]\Big\}\\
&\le \left(\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\left\|A_{\theta_i(k)}\tilde{x}_i(k)\right\|^2\right)^{\frac{1}{2}} \times\left(\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\left\|H_{\theta_i(k)}[g(k)-x^{(N)}(k)]\right\|^2\right)^{\frac{1}{2}}\\
&\le \left(\sum_{j=1}^{m}tr(A_jZ_jA_j^T)\max_{1\le j\le m}\|H_j\|^2 O\left(\frac{1}{N}\right)\right)^{\frac{1}{2}} = O\left(\frac{1}{\sqrt{N}}\right),
\end{aligned}$$

by (22) and (32) and Theorem 4.2, we can get (31). □

Remark 4.1. Comparing (31) with (5), the above index value is larger than the optimal one (with full state information and centralized control) by a term of $\sum_{j=1}^{m}tr(A_jZ_jA_j^T)+O(1/\sqrt{N})$. Thus, the distributed control (15) is sub-optimal, and from (21) the error between the sub-optimal index value and the optimal one in the case of large population is mainly attributed to the partial observation of the system states.

Below we show the equilibrium property of (15).

Definition 4.4. A set of control $\{u_i \in \mathcal{U}_{l,i}, 1\le i\le N\}$ is called an ε-Nash equilibrium if there exists $\varepsilon \ge 0$ such that for any $1\le i\le N$,

$$J_i^N(u_i,u_{-i}) \le \inf_{u_i'\in\mathcal{U}_{g,i}} J_i^N(u_i',u_{-i})+\varepsilon,$$

where $u_{-i}=\{u_0,\dots,u_{i-1},u_{i+1},\dots,u_N\}$.

Theorem 4.5. *For (1)–(3), if (A1)–(A4) hold, then the distributed control (15) is an ε-Nash equilibrium, where $\varepsilon=\sum_{j=1}^{m}tr(A_jZ_jA_j^T)+O(1/\sqrt{N})$. Particularly, $\varepsilon\to 0$ when $N\to\infty$ and one of the following conditions holds:*

(a) $L_j=0$, C_j *is invertible for* $j=1,\dots,m$;
(b) $L_j=0$, $\|D_j\|\to 0$ *for* $j=1,\dots,m$.

Proof. From Theorems 3.1 and 4.3, it follows that (15) is an ε-Nash equilibrium, where $\varepsilon=\sum_{j=1}^{m}tr(A_jZ_jA_j^T)+O(1/\sqrt{N})$. When $L_j=0$ and C_j is invertible, we have $Z_j=0$, which implies $\varepsilon\to 0$ as $N\to\infty$. From $L_j=0$, (13) can be transformed into $Y_k=\sum_{j=1}^{m}p_{jk}A_j(I-M_jC_j)Y_j(I-C_j^TM_j^T)A_j^T+\pi_jD_jD_j^T$, where $k=1,\dots,m$. From Costa et al. (2005) with (A2), it follows that $Y=\sum_{n=0}^{\infty}\mathcal{T}^n(\pi DD^T)<\infty$, where $Y=(Y_1,\dots,Y_m)$, $\pi DD^T=(\pi_1D_1D_1^T,\dots,\pi_mD_mD_m^T)$, and for any $V=(V_1,\dots,V_m)$, $\mathcal{T}_k(V)=\sum_{j=1}^{m}p_{jk}A_j(I-M_jC_j)V_j(I-C_j^TM_j^T)A_j^T$. Thus, as $\|D_j\|\to 0$, $\|Y_j\|\to 0$ for $j=1,\dots,m$, which gives $\varepsilon\to 0$, as $N\to\infty$. □

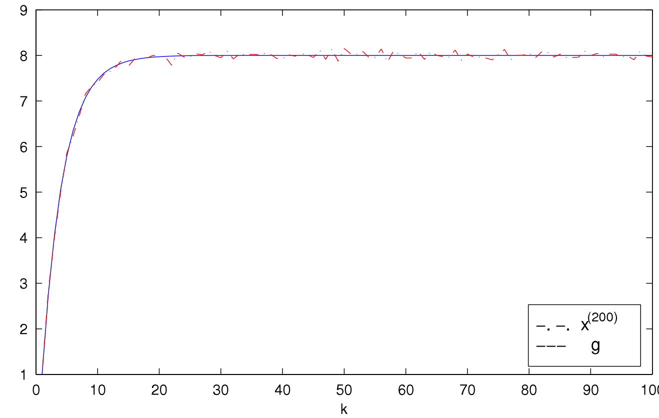

Fig. 1. Trajectories of $x^{(200)}$ and g.

5. Numerical simulation

We now use a numerical example to illustrate the consistency of the MF estimation and the sub-optimality of distributed control laws.

For the system (1)–(2), we set the parameters as follows: $A_1=-1, A_2=-0.8, D_1=1, D_2=0.6, C_1=1, C_2=1, L_1=0.6, L_2=0.4$. $\{\theta_i(k)\}$ is a Markov chain taking value in $\{1,2\}$ with the transition probability matrix $P=\begin{pmatrix}0.75 & 0.25\\ 0.25 & 0.75\end{pmatrix}$, and the stationary distribution (0.5 0.5). Both $\{w_i(k), 1\le i\le N\}$ and $\{v_i(k), 1\le i\le N\}$ are Gaussian white noise sequences with the normal distribution $N(0,1)$. Let $\{x_{i0}, i=1,\dots,N\}$ be independent and identically distributed random variables with $N(1,0.8)$. For the index (3), the parameters are set as $H_1=0.9, H_2=0.6, \alpha_1=1, \alpha_2=3$.

Noticing that $A=(-1,-0.8), D=(1,0.6), C=(1,1)$, and $\sum_{j=1}^{2}\pi_jH_j=0.75<1$, it can be verified that, Assumptions (A1)–(A4) hold. From (15) we get the following distributed control laws: for $1\le i\le N$,

$$u_i(k)=H_{\theta_i(k)}g(k)+\alpha_{\theta_i(k)}-A_{\theta_i(k)}\hat{x}_i(k), \quad (33)$$

where $M_1=0.7469, M_2=0.8022$, $\hat{x}_i(k)$ given by (11), $\hat{x}_i(0)=1$, and $g(k+1)=0.75g(k)+2$.

First, we check the consistency of the MF estimation. Letting the number of the agents be 200, the trajectories of $x^{(N)}$ and g are shown as Fig. 1. It can be seen that, g almost coincides with $x^{(200)}$. This illustrates the consistency of the MF estimate given by Theorem 4.2.

We now consider the index values of all the agents under the distributed control (33). Let $J^N=\max_{1\le i\le N}J_i^N$. Then, from (13) we have $Z_1=0.1344$ and $Z_2=0.0411$, and thus, by Theorem 4.3 J^N should converge to $\sum_{j=1}^{2}\pi_j tr(D_jD_j^T)+\sum_{j=1}^{2}tr(A_jZ_jA_j^T)=0.8555$. When the number of agents grows from 1 to 200, the trajectories of J^N are shown in Fig. 2, from which one can see that the index value tends to 0.8555.

6. Concluding remarks

In this paper, we study distributed output feedback control of Markov jump MASs under the game framework. Each agent can only get its noisy output and jump parameters. We design distributed control and show its sub-optimality.

It is worth pointing out that, the MAS considered in the paper is a type of individual–population interacting system, in which the population effect on a given agent is nearly deterministic and the single-agent impact is negligible for the large population case. For the distributed control of this type of systems, there are many interesting problems worth investigating, such as the case with unknown jump parameters, and the case where agents are coupled

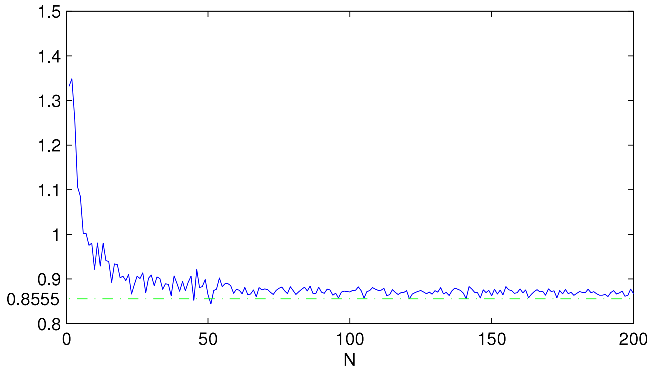

Fig. 2. Trajectory of J^N with respect to N.

by both dynamics and cost indices. For the latter case, the filtering equations of agents are coupled by the MF estimation function, which together with coupled filtering Riccati equations will bring essential difficulty into the analysis of the closed-loop system.

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